STRONG CONVERGENCE OF THE ITERATES OF AN OPERATOR†

BY

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ABSTRACT

Let $R = \int \phi(t) P_t dt$ where P_t is a semi group of operators. Conditions are established for the strong convergence of R^n.

Let X be a Banach space, X^* its dual, and P_t, $0 \leqq t$, a strongly continuous semi group of operators with $\| P_t \| \leqq 1$ and $P_0 = I$. Let $0 < \phi(t)$ be a continuous function with $\int_0^\infty \phi(t)dt = 1$ and define $R = \int_0^\infty \phi(t)P_t dt$. (If $\phi(t) = \lambda e^{-\lambda t}$ for $\lambda > 0$ then $(1/\lambda)R$ is $(\lambda I - A)^{-1}$ where A is the infinitesimal generator.)

Let $0 \leqq t_0$ and $0 < h$ and set

$$Q_1 = \int_{t_0}^{t_0+h} \phi(t)P_t dt \cdot \left(\int_{t_0}^{t_0+h} \phi(t)dt \right)^{-1} \text{ and}$$

$$Q_2 = \left(\int_0^{t_0} + \int_{t_0+h}^\infty \right) \phi(t)P_t dt \cdot \left(1 - \int_{t_0}^{t_0+h} \phi(t)dt \right)^{-1}.$$

Then $\| Q_1 \| \leqq 1$ $\| Q_2 \| \leqq 1$, $Q_1Q_2 = Q_2Q_1$ and

$$R = \left(\int_{t_0}^{t_0+h} \phi(t)dt \right) Q_1 + \left(1 - \int_{t_0}^{t_0+h} \phi(t)dt \right) Q_2.$$

By [1, Lem. 2.1.]:

$$\| R^n(Q_1 - Q_2)x \| \to 0 \text{ for all } x \in X.$$

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Thus $\| R^n y \| \to 0$ provided y is in the space X_1, generated by $(Q_1 - Q_2)x$ where x, t_0, and h are arbitrary. Let us use the Hahn-Banach theorem which states: if $x_0^* \in X^*$ and $x_0^*(X_1) = 0$ then

$$0 = x_0^*(Q_1 - Q_2)x \underset{h \to 0}{\to} x_0^*(P_{t_0}x - Rx).$$

Thus $P_{t_0}^* x_0^* = R^* x_0^*$ for every $0 \leqq t_0$ or $P_{t_0}^* x_0^* = x_0^*$ for every $0 \leqq t_0$.

Thus we have proved Theorem 1.

THEOREM 1. *Let P_t be a strongly continuous semi group of contractions and $P_0 = I$. If $\phi(t)$ is a positive continuous function on $[0, \infty)$ and $P_0 = I$, $R = \int_0^\infty \phi(t) P_t dt$ then $\| R^n x \| \underset{h \to \infty}{\to} 0$ for every $x \in X$ such that $x_0^*(x) = 0$ for all x_0^* with $P_t^* x_0^* \equiv x_0^*$.*

REMARK. In [2] it was proved that $P_t^* x^* = x^*$ for all t if and only if $R^* x^* = x^*$ by a similar technique. Using more detailed analysis one can show that $\phi(t)$ does not have to be strictly positive or continuous. (If R is given by ϕ, R^n is given by convolutions of ϕ and these convolutions become smoother as n increases.)

If we assume that X is a reflexive space, a more precise result can be established. If $Rx_0 = x_0$ consider the convex compact set

$$\{x_0^* : \| x_0^* \| = 1,\ x_0^*(x_0) = 1\}.$$

It is invariant under R^* and thus contains a fixed point. There exists a functional x_0^* with $\| x_0^* \| = x_0^*(x_0) = 1$, $R^* x_0^* = x_0^*$.

By reflexivity, if $R^* x_0^* = x_0^*$, there exists a vector x_0 with $\| x_0 \| = x_0^*(x_0) = 1$ and $Rx_0 = x_0$. Define

$$K = \{x : Rx = x\},\ L = \{x : x^*(x) = 0 \text{ for all } x^* \text{ with } R^* x^* = x^*\}.$$

By [2, Lem. 2] and Theorem 1, if $y \in L$ then $\| R^n y \| \to 0$. Hence if $z = x + y$ for $x \in K$ and $y \in L$ then

$$\| z \| \geqq \| R^n z \| \to \| x \|.$$

Or $Ez = x$ is a bounded operator of norm 1. To see that $K + L$ is the entire space, let $x^*(K + L) = 0$. Since $x^*(L) = 0$, then $R^* x^* = x^*$, by the Hahn-Banach theorem. Following from the remarks above, there exists a vector x with $Rx = x$ and $x^*(x) = 1$ which contradicts $x^*(K) = 0$. Thus we have proved Theorem 2.

THEOREM 2. *Assume the conditions of Theorem 1 and let X be reflexive. Then R^n converges strongly to a projection on the space $\{x: Rx = x\}$.*

REFERENCES

1. S. R. Foguel and B. Weiss, *On convex power series of a conservative Markov operator*, (to be published in the Proc. Amer. Math. Soc.).

2. M. Falkowitz, *On finite invariant measures for Markov operators* (to be published).

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